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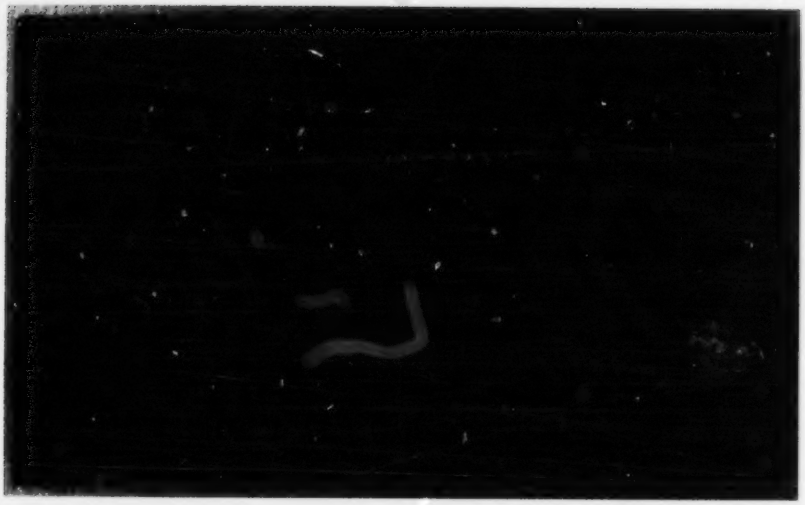

CENTRE DE RECHERCHE SUR LES TRANSPORTS
UNIVERSITÉ DE MONTRÉAL
C.P. 6128, SUCCURSALE A,
MONTRÉAL
CANADA H3C 3J7

TÉLÉPHONE: (514) 343-7575
TÉLEX: (21) 05-561325
EMAIL: crt@crt.umontreal.ca
TÉLÉCOPIEUR (FAX): (514) 343-7121

92-02313/

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Designing Collection Routes Through Bank Branches

by
Véronique Lambert[1,2]
Gilbert Laporte[2]
François Louveaux[1]

[1] Facultés Universitaires Notre-Dame de la Paix, 8 Rempart de la Vierge, B-5000 Namur, Belgium.
[2] Centre de recherche sur les transports, Université de Montréal, C.P. 6128, succursale A, Montréal, Québec, Canada H3C 3J7.

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Abstract

Most banks must send out vehicles on a regular basis to their branches in order to collect cash and negotiables left by depositors. The problem is modeled as a vehicle routing problem with constraints and objective terms specific to the particular banking context, such as penalties for lost interest. Two versions of the problem are studied: the deterministic case, and the stochastic case in which travel times are random. These two versions are modeled as integer mathematical programs, and solved by means of a heuristic procedure. Problems derived from a real situation (a Belgian bank network) are solved under different scenarios.

Keywords: Vehicle routing, integer stochastic programming.

Résumé

La plupart des banques doivent envoyer à intervalles réguliers des véhicules recueillir les fonds déposés dans les différentes succursales. On modélise ce problème comme un problème de tournées de véhicules, avec des contraintes et un objectif propres au contexte bancaire. En particulier on doit tenir compte des pénalités engendrées par l'intérêt perdu. Deux versions du problème sont étudiées: le cas déterministe et le cas stochastique avec durées de trajet aléatoires. On modélise ces deux problèmes par la programmation mathématique en nombre entiers et on les résout à l'aide d'une procédure heuristique. Des problèmes dérivés d'une situation réelle (un réseau bancaire belge) sont ensuite résolus selon divers scénarios.

Mots-clés: Problèmes de tournées de véhicules, programmation stochastique en nombres entiers.

1. Introduction

Most banks must send out vehicles on a regular basis to their branches in order to collect cash and negotiables left by depositors and take them to a central office. This operation is usually carried out daily by the bank's own vehicles or by a contractor. The object of this paper is to analyze vehicle route planning in this context.

Our problem belongs to the family of *Vehicle Routing Problems* (VRPs). These have been extensively studied in the operations research literature: see, for example, the surveys of Bodin, Golden, Assad and Ball [1], Christofides [2], Laporte and Nobert [13] as well as the book edited by Golden and Assad [8]. In essence, VRPs consist of designing minimum cost routes

(i) starting and ending at a common depot (central office);

(ii) visiting every customer (branch);

(iii) satisfying a number of side constraints.

Here, the only side constraint is that all vehicles must return to the central office before a preset deadline. In classical VRPs, routing cost is defined as a weighted sum of vehicle fixed costs and mileage costs. Our problem contains an additional cost component: money accumulated at a branch between the vehicle's visit and closing time is not collected until the next day and therefore loses one day's interest. This has a direct impact on the cost and shape of the solution since it may be preferable to delay as much as possible visits to branches producing a large amount of cash. Penalties associated with lost interest are directly taken into account in the objective function.

Vehicle capacity is always sufficiently large to make all collections and is not a binding factor. However, the total cash amount carried by a vehicle at any time should not exceed a given threshold imposed by insurance companies. But since this threshold is only rarely exceeded in practice, it is left out of our analysis.

Most VRP studies reported in the O.R. literature implicitly assume that all problem parameters are deterministic. This is clearly not always the case in practice. For example, the amounts collected at the various branches fluctuate from day to day and are random variables. Travel times are also random since they depend on traffic conditions. This affects cost since when a vehicle arrives at the central office after the deadline, all the money it carries is credited to the next day. We have chosen in this study to ignore the daily variations in the amounts collected at the branches, and to use instead their average values. Travel times are considered as deterministic in one version of the problem and as stochastic in another version. For recent references on stochastic VRPs, see Laporte, Louveaux and Mercure [11], Laporte and Louveaux [10], and Dror, Laporte and Trudeau [7].

The remainder of this paper is organized as follows. In the next two sections, we describe two integer linear programming models for the problem: one for the deterministic case, and one for the stochastic case. We then show in Section 4 how these models can be solved by adapting the Clarke and Wright algorithm [3] originally designed for the classical VRP. In Section 5, computational tests and sensitivity analyses on both models are carried out on two data sets derived from the branch network of a bank operating in the French-speaking part of Belgium. The conclusion

follows in Section 6. Finally, a glossary of the main notation used in this paper is provided in the Appendix.

2. Deterministic Model

The problem is defined on a graph $G = (V, A)$, where $V = \{0, 1, ..., n\}$ is a set of vertices and A is a set of arcs. Vertex 0 represents the depot or central office, while the remaining vertices correspond to bank branches. With each arc $(i, j) \in A$ is associated a distance d_{ij}, a travel duration t_{ij} and a travel cost c_{ij}. The three matrices $(d_{ij}), (t_{ij})$ and (c_{ij}) satisfy the triangle inequality, and both t_{ij} and c_{ij} are linearly related to d_{ij}. The duration of service at branch $i \in V$ is equal to u_i. Denote by T the maximal allowed duration of any vehicle route and by f the fixed daily cost of a vehicle. We assume the problem is defined on a daily basis, that everyday operations start at time 0 and, for the sake of simplicity, we ignore week-ends and statutory holidays. Hence, T is the latest arrival time of a vehicle at the depot. Branch i opens at time r_i and closes at time s_i. Since it is often impractical for a branch to be visited right at closing time, an upper bound v_i $(\leq s_i)$ on the arrival time of the collection vehicle is also imposed. Let a_i denote the (average) daily sum collected at branch i. It is assumed that money accumulates at a constant rate over opening hours. Only money arriving at the central office before some cut-off time $T' \geq T$ is credited on the same day; otherwise, it loses one day's interest. The reason why we impose $T' \geq T$ is that it makes little managerial senses to plan vehicle routes that systematically arrive at the depot after T'. Let α be the unit penalty for daily lost interest and finally, let M be an arbitrarily large constant.

Variables for this problem are defined as follows: y_i is a variable equal to the arrival time of the vehicle at branch i, x_{ij} is a binary variable equal to 1 if and only if arc (i,j) is used in the solution, and m is the number of vehicles. Note that r_i, s_i, u_i and v_i are known constants. The *deterministic model* is then

$$\text{minimize} \quad fm + \sum_{\substack{i,j=0 \\ i \neq j}}^{n} c_{ij}x_{ij} + \alpha \sum_{i=1}^{n} a_i \left[\frac{\min\{T', v_i\} - y_i}{s_i - r_i} \right]$$

subject to

$$\sum_{i=1}^{n} x_{i0} = \sum_{j=1}^{n} x_{0j} = m \tag{1}$$

$$\sum_{\substack{i=1 \\ k \neq i}}^{n} x_{ik} = \sum_{\substack{j=1 \\ k \neq j}}^{n} x_{kj} = 1 \qquad (k = 1, \ldots, n) \tag{2}$$

$$y_i - y_j + Mx_{ij} \leq M - t_{ij} - u_i \qquad (i, j = 1, \ldots, n; i \neq j) \tag{3}$$

$$\max\{t_{0i}, r_i\} \leq y_i \leq \min\{T - u_i - t_{i0}, v_i\} \qquad (i = 1, \ldots, n) \tag{4}$$

$$x_{ij} = 0 \text{ or } 1 \qquad (i, j = 0, \ldots, n; i \neq j) \tag{5}$$

$$m \geq 1. \tag{6}$$

In this formulation, the three terms of the objective function represent vehicle fixed costs, routing costs, and "indirect penalty" for lost interest, respectively. The factor within brackets is equal to the fraction of the money accumulated at branch i that is subject to penalty. If $v_i \geq T'$, all proceeds collected between the arrival of the vehicle (at time y_i) and the cut-off time T' lose one day's interest. Money accumulated between T' and v_i also loses interest, but this is an unavoidable loss which cannot be reduced through optimization. If $v_i < T'$, the penalty applies to all proceeds received between y_i and v_i. Note that the objective is linear since in the bracketed expression, only y_i is a variable. Extensions of the model to accommodate non-linear money accumulation patterns are possible, but only at the cost of increased algorithmic difficulty.

Constraints (1) and (2) are degree constraints. Note that m can be treated as a variable, or as a constant when the size of the fleet is known. In this case, the first term of the objective function is unnecessary. Constraints (3) play three different roles: (i) they ensure that when j immediately follows i on a route, then $y_j \geq y_i + (u_i + t_{ij})$; (ii) they prevent the formation of subtours disconnected from the depot and (iii) they prohibit routes of total duration in excess of T. These constraints can be strengthened: using a lifting procedure, Desrochers and Laporte [5] show that it is valid to add an extra term equal to $(M - t_{ij} - t_{ji})x_{ji}$ to their left-hand side. Constraints (4) ensure that the arrival time of a vehicle at a branch occurs when the branch is open between r_i and s_i, while allowing sufficient time for driving from and to the depot. Finally, constraints (5) and (6) impose integrality and non-negativity conditions on the variables.

3. Stochastic Model

In the *stochastic model* it is assumed that some arcs of A are congested with probability p. In practice, congestion is more likely to occur at rush hour, in downtown areas, or on roads leaving or entering major towns. Let $A' \subseteq A$ be the set of arcs subject to congestion and assume that for every arc $(i,j) \notin A', t_{ij} = \beta_0 d_{ij}$ where β_0 is the inverse of vehicle speed; for every arc $(i,j) \in A', t_{ij}$ is equal to $\beta_1 d_{ij}$ with probability $1-p$ and to $\beta_2 d_{ij}$ with probability p, where $\beta_2 > \beta_1$.

In such a context, it is important to make a distinction between two classes of decisions: (i) *first stage decisions* are those taken before knowledge of the actual "states of nature", i.e. congestion levels; (ii) *second stage decisions* are only taken once the outcome of the random variables is known.

In our problem, it makes sense to define first stage decisions as the *a priori* planned routes. Second stage decisions then correspond to the observation of the actual visiting times y_i and to the computation of the associated costs. It is not realistic here to modify planned routes when congestion levels become known, since this would involve notifying the various branches of the new (expected) schedule and this could have an impact on the accounting operations at branch level.

The stochastic model can be derived from the deterministic model as follows. That part of the model which depends on first stage decisions only (m and the x_{ij} variables) remains unchanged. This includes constraints (1), (2), (5) and (6) as well as the first two terms of the objective function. The modifications to the deterministic model will be presented in three steps: (i) constraints (3) and (4); (ii)

the "indirect penalty" term of the objective function and (iii) a fourth term in the objective associated with possible late arrival at the depot.

3.1. Modified Constraints (3) and (4)

Even if travel times are stochastic, it is necessary to ensure that actual visiting times *always* take place within visiting hours. We also impose a constraint on the maximum *expected duration* of any vehicle route. These two constraints can be defined in terms of the *expected travel time* $\bar{t}_{ij}$ on arc (i,j) and of the *expected arrival time* $\bar{y}_i$ at branch i. Expected travel times are computed as

$$\bar{t}_{ij} = \begin{cases} \beta_0 d_{ij} & \text{if } (i,j) \notin A' \\ (1-p)\beta_1 d_{ij} + p\beta_2 d_{ij} & \text{if } (i,j) \in A'. \end{cases} \tag{7}$$

Constraints on expected route duration can now be imposed by replacing t_{ij} by $\bar{t}_{ij}$ and y_i by $\bar{y}_i$ in (3).

We now wish to impose conditions on $\bar{y}_i$ so that no matter what the congestion levels are, actual visiting times are always feasible. In order to modify constraints (4) to that effect, define for every arc $(i,j) \in A'$: $\min(t_{ij}) = \beta_1 d_{ij}$ and $\max(t_{ij}) = \beta_2 d_{ij}$; for $(i,j) \notin A'$, let $\min(t_{ij}) = \max(t_{ij}) = \beta_0 d_{ij}$. Then

$$\bar{t}_{ij} / \min(t_{ij}) = \begin{cases} (1-p) + p\beta_2/\beta_1 \geq 1 & \text{if } (i,j) \in A' \\ 1 & \text{if}(i,j) \notin A' \end{cases}$$

and

$$\bar{t}_{ij} / \max(t_{ij}) = \begin{cases} (1-p)\beta_1/\beta_2 + p \leq 1 & \text{if } (i,j) \in A' \\ 1 & \text{if}(i,j) \notin A'. \end{cases}$$

Similarly, define $\min(y_i)$ and $\max(y_i)$, the earliest and latest possible arrival times at branch i. Under worst case scenarios, the ratio $\overline{y}_i/\min(y_i)$ is equal to $(1-p)+p\beta_2/\beta_1$, and the ratio $\overline{y}_i/\max(y_i)$ is equal to $(1-p)\beta_1/\beta_2+p$, so that constraints (4) can be rewritten as

$$\max\left\{\overline{t}_{0i}, r_i(1-p+p\beta_2/\beta_1)\right\} \leq \overline{y}_i \leq \min\left\{T-u_i-\overline{t}_{i0}, v_i[(1-p)\beta_1/\beta_2+p]\right\} \ (i=1,...,n). \qquad (4')$$

3.2. Modified indirect penalty term of the objective function

We are now interested in the *expected indirect penalty* at every branch i. Because of the linear relationship between this penalty and the arrival time y_i, it is valid to merely replace y_i by its expected value $\overline{y}_i$ in the objective function.

3.3. "Direct penalty" for late arrival at the depot

Constraints (4') guarantee that even in worst case scenarios, all branches are always visited within opening hours and all vehicles arrive at the depot before time T. Since we have assumed $T \leq T'$, it follows that all money contained in these vehicles will be credited on the same day.

A less constraining scenario occurs when $T' < T \leq \min\{v_i\}$ and it is required that all collections take place within branch visiting hours. Here, return at the depot before T' is no longer mandatory. Then the right-hand side of (4') is simply replaced by $v_i[(1-p)\beta_1/\beta_2+p]$. Moreover, the possibility of losing one day's interest on the whole vehicle content must now be taken into account in the objective function. For vehicle route ℓ, let $\overline{b}_\ell$ denote the expected value of the vehicle content given that

route duration exceeds T'. Then the objective function must include a fourth term equal to $\alpha \sum_{\ell} \bar{b}_{\ell}$.

Since our model does not explicitly identify vehicles, i.e. setting $x_{ij} = 1$ does not indicate which vehicle travels on arc (i,j), the extra term just added to the objective function cannot be expressed in terms of the original variables of the model. Developing a more complex model to handle this case would be cumbersome and is quite unnecessary. Indeed, the new penalty term can be computed quite easily for any feasible solution. We now illustrate this computation for the case of mutually independent states.

Let R_ℓ be the set of arcs on route ℓ : $R_\ell = \{(i_0,i_1),(i_1,i_2),...,(i_{h-1},i_h),(i_h,i_{h+1})\}$, where h depends on ℓ and $i_0 = i_{h+1} = 0$. Arc $(i,j) \in R_\ell$ is in state $\xi_{ij} = 0$ if it is not congested and in state $\xi_{ij} = 1$ otherwise. Let $\xi = (\xi_{ij})$ be a state vector associated with R_ℓ and define for every $(i,j) \in R_\ell$

$$t_{ij}^{\xi} = \begin{cases} \beta_0 d_{ij} & \text{if } (i,j) \notin A' \\ \beta_1 d_{ij} & \text{if } (i,j) \in A' \text{and } \xi_{ij} = 0 \\ \beta_2 d_{ij} & \text{if } (i,j) \in A' \text{and } \xi_{ij} = 1 \end{cases}$$

and

$$p_{ij}^{\xi} = \begin{cases} 1 & \text{if } (i,j) \notin A' \\ 1-p & \text{if } (i,j) \in A' \text{and } \xi_{ij} = 0 \\ p & \text{if } (i,j) \in A' \text{and } \xi_{ij} = 1. \end{cases}$$

The amount collected at any branch i_k is a function of ξ and of the state ξ' the previous day, but in order to simplify matters, we assume that it is always equal to its expected value a_{i_k}. Now let $H_\ell = \left\{ \xi : \sum_{k=0}^{h} t_{i_k i_{k+1}}^{\xi} \geq T' \right\}$. Then

$$\bar{b}_\ell = \sum_{\xi \in H_\ell} \left(\prod_{k=0}^{h} p^{\xi}_{i_k i_{k+1}} \right) \left(\sum_{k=1}^{h} a_{i_k} \right). \tag{8}$$

This summation extends over all states ξ for which the duration of route ℓ exceeds T'. The first factor is the probability that state ξ occurs, whereas the second factor is the average amount of money carried by the vehicle. The computation of $\bar{b}_\ell$ is exponential in the number of "random arcs" on a route. However, this number is always small in practice (usually less than 4) and this poses no computational difficulty.

4. Algorithm

The deterministic model just described is a modified version of the vehicle routing with time windows for which a number of column generation techniques have been developed (see, for example, Desrosiers, Sauvé and Soumis [6] and Desrochers, Desrosiers and Solomon [4]). These algorithms perform rather well on tightly constrained moderate size problems, but less well on problems with large time windows, as is the case in our application. Moreover, column generation algorithms are less competitive in problems where customer locations are randomly distributed in the plane (as seems to be the case in our problem) and in contexts where several customers are visited on the same route as degeneracy problems are then more frequent (Desrochers, Desrosiers and Solomon [4]). One additional drawback of exact algorithms in the context of our application is their high unpredictability: they can be very efficient on a particular problem and fail on the next. Finally, none of the known exact algorithms for VRPs can be easily adapted to the stochastic case.

Our stochastic model is a particular version of VRPs with stochastic travel times (Laporte, Louveaux and Mercure [12]). Although it could conceivably be solved by means of the integer L-shaped procedure proposed by Laporte and Louveaux [10], the size of our application would in all likelihood make this approach computationally difficult.

We have opted for a heuristic algorithm that can easily be adapted to the problem under consideration and is also robust and efficient, the Clarke and Wright algorithm. This algorithm was originally designed to minimize functions consisting of the sum of routing costs in VRPs, i.e. functions of the form $z = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} c_{ij}x_{ij}$. It can be summarized as follows.

Step 1. For every arc (i,j) such that $i,j \geq 1$, compute the *saving* $s_{ij} = c_{i0} + c_{0j} - c_{ij}$.

Step 2. Initially create n routes, each including the depot and one vertex of $V \setminus \{0\}$.

Step 3. Consider two vehicle routes containing arcs $(i,0)$ and $(0,j)$ respectively. If $s_{ij} > 0$, tentalively merge these two routes by introducing arc (i,j) and by deleting arcs $(i,0)$ and $(0,j)$. Implement the merge if the resulting route is feasible. Repeat this step until no further improvement is possible. Stop. ■

This procedure implicitly ignores vehicle fixed costs and fleet size. Vehicle costs can easily be taken into account by adding a constant f to every $c_{0j}(j \geq 1)$. Solutions with a fixed number of vehicles can be obtained by repeating Step 3 until the required number of routes has been reached, even if the savings become negative.

There exist several variants and implementations of this procedure (see Paessens [15]). We have implemented the "parallel" version of the algorithm which consists

of selecting the two nodes i and j to be linked according to the largest saving that can be generated.

This simple method can easily be extended to more general objective functions that do not simply minimize routing costs, but include other terms as well (such as penalties or expected penalties for lost interest). This is the case of both our deterministic and our stochastic models. Consider two vehicle routes ℓ_1 and ℓ_2 including arcs $(i,0)$ and $(0,j)$ respectively, and the costs $z(\ell_1)$ and $z(\ell_2)$ of these routes. Merging ℓ_1 and ℓ_2 by introducing (i,j) and deleting $(i,0)$ and $(0,j)$ yields a route ℓ of cost $z(\ell)$. If ℓ is feasible and the saving $s_{ij} = z(\ell_1) + z(\ell_2) - z(\ell)$ is positive, the merge is then implemented. This method can be quite time consuming since it requires updating the savings at each iteration. Computations can be eased, however, by using appropriate data structures (Paessens [15]) and heap-sorting methods (Nelson, Nygard, Griffin and Schreve [14]). Both these features were included in our implementation of the algorithm. The use of heap structures [14] allows significant computing time and memory savings. In the deterministic case, the time complexity of the Clarke and Wright algorithm is reduced from $O(n^2 \log n)$ [1] to $O(n \log n)$ [14]. In the stochastic case, the time complexity increases to $O(2^a n \log n)$ (where $a = |A'|$), since $\overline{b}_\ell$ has to be computed. Our program was coded in Turbo Pascal 4 and implemented on an Olivetti M380 micro-computer.

5. Computational Results

The above algorithm was tested on two data sets derived from a practical problem, both for the deterministic and stochastic cases.

We used the branch network of a bank operating in the French-speaking part of Belgium, with headquarters at Namur (used as the "depot") – see Figure 1. The number of branches is equal to 28 in the first data set and to 44 in the second case. The branch names for each problem are given in Table 1. The distance matrix (d_{ij}) was computed using the charts compiled by Houet and Cleeren [9]. Travel costs c_{ij} were estimated at 15 Belgian francs (BEF) per kilometer. The cost of using a vehicle is estimated to be in excess of 10 000 BEF per day.

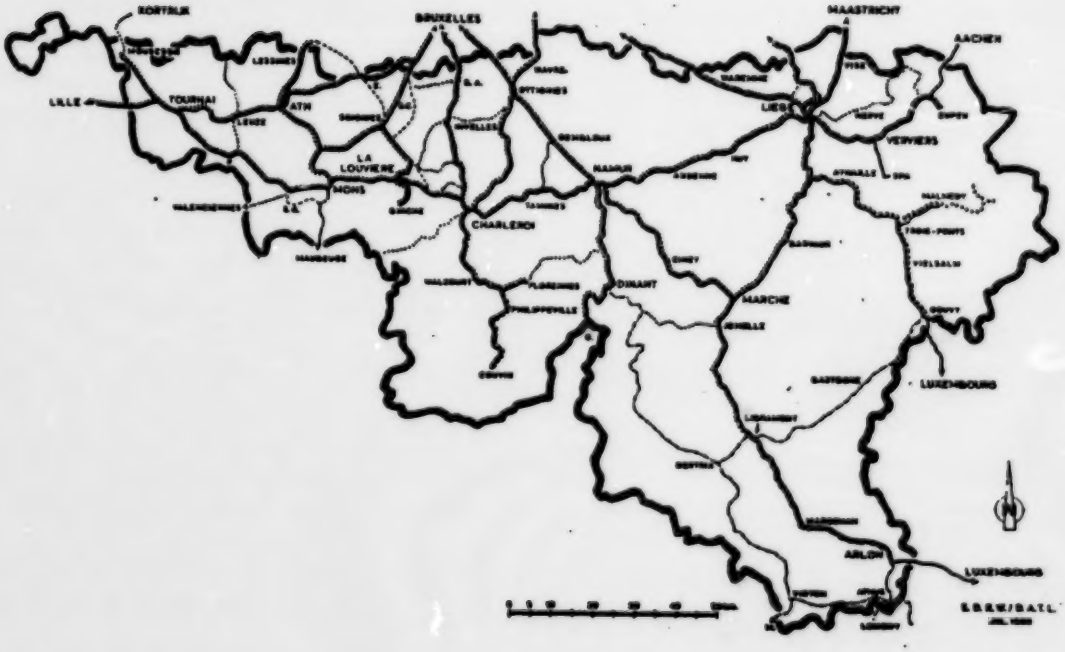

Figure 1. Map of French - speaking Belgium

Table 1. Branch locations

Depot : Namur

Branches common to 28 city and 44 city problems

Anhée	Floreffe	Jodoigne	Profondeville
Auvelais	Forville	Marbais	Rhisnes
Bois-de-Villers	Fosses	Mettet	Saint-Servais
Bouge	Gembloux	Moustier	Salzinnes
Eghezée	Jambes	Namèche	Sombreffe
Erpent	Jauche	Orp	Tamines
Flawinne	Jemeppe	Perwez	Walhain-St-Paul

Additional branches for 44 city problem

Assesse	Bouillon	Hastière	La Roche
Barvaux	Ciney	Havelange	Marche
Beauraing	Dinant	Hotton	Rochefort
Bièvre	Forrières	Jemelle	Wellin

All branches open at 9 a.m., close at 4 p.m. and must be visited before 2 p.m., so that $r_i = 0, s_i = 7$ and $v_i = 5$ for all i. Also the values of T and T' are equal to 5 hours. The service time at any branch is equal to 5 minutes. Data on the daily amounts collected at the various branches are confidential, so we had to use estimates based on typical values, by assuming that a_i is proportional to the population of city i. The daily penalty factor for lost interest was taken as 10/365 percent or 0.000274.

In deterministic problems, we executed tests with two different speeds: 25 km/h and 40km/h. In stochastic problems, we defined A' as the set of arcs incident to

one of the following cities: Namur, Dinant, Gembloux and Rochefort in 44 city problems, and Namur, Bouge, Gembloux, in 28 city problems, so that $|A'| = 176$ in the first case and $|A'| = 84$ in the second case. These cities were chosen since they are those around which congestion is more likely. The value of p was set at 0.3 and the values of β_0, β_1 and β_2 were: $\beta_0 = 1/25, \beta_1 = 1/28, \beta_2 = 1/19$ or $\beta_0 = 1/40, \beta_1 = 1/47, \beta_2 = 1/26$.

The number m of vehicles was fixed at different values for various runs, so that their fixed costs were ignored. Comparing the costs of the solutions obtained for different values of m may help decide whether it is better to use a larger or a smaller number of vehicles.

The algorithm was first applied to deterministic problems with the following parameters:

1)	$n = 28,$	$m = 5, 4, 3,$	$\beta_0 = 1/25,$	$\alpha = 0.000274;$
2)	$n = 28,$	$m = 5, 3,$	$\beta_0 = 1/40,$	$\alpha = 0.000274;$
3)	$n = 28,$	$m = 3, 2,$	$\beta_0 = 1/25,$	$\alpha = 0;$
4)	$n = 44,$	$m = 6, 5,$	$\beta_0 = 1/40,$	$\alpha = 0.000274;$
5)	$n = 44,$	$m = 5, 4,$	$\beta_0 = 1/40,$	$\alpha = 0.$

The main results are reported in Table 2. These provide a good indication on the best number of vehicles to use. For example, in the first test series, going from 5 to 4 vehicles generates routes of reasonable length while adding very little to the total cost. Our algorithm failed to generate a solution with 3 vehicles (however, when α was set equal to 0, a feasible solution was found – see test series #3).

Table 2. Computational results for deterministic problems

Test series	n	m	β_0	α	Duration of shortest route (minutes)	Duration of longest route (minutes)	Routing cost (BEF)	Lost interest due to early collections	Total cost (BEF)	Computation time (seconds)
1	28	5	1/25	0.000274	36	273	4650	2485	7135	82
		4		0.000274	167	273	4650	2704	7354	138
		3		0.000274		>300				
2	28	5	1/40	0.000274	27	186	4635	1773	6408	83
		3		0.000274	144	238	4275	2845	7120	130
3	28	3	1/25	0	279	299	4590	0	4590	328
		2		0		>300				
4	44	6	1/40	0.000274	144	300	11340	4130	15470	1192
		5		0.000274		>300				
5	44	5	1/40	0	173	300	10800	0	10800	3280
		4		0		>300				

Additional test series were also executed for stochastic problems with the following parameters:

1) $n = 28, \quad m = 8, 6, 5, \quad \beta_0 = 1/25, \quad \beta_1 = 1/28, \quad \beta_2 = 1/19, \quad \alpha = 0.000274;$
2) $n = 44, \quad m = 12, 9, 6 \quad \beta_0 = 1/40, \quad \beta_1 = 1/47, \quad \beta_2 = 1/26, \quad \alpha = 0.000274.$

The main results are summarized in Table 3. As for deterministic problems, this table enables comparisons between solutions containing different numbers of vehicles. Attempts to solve the problem with fewer vehicles usually resulted in infeasible solutions. When $n = 28$, the best number of vehicles seems to be 5. The longest route in this solution has an expected duration of 230 minutes and a zero probability of lasting more than 300 minutes. It has a slightly larger expected

variable cost than the solution with 6 vehicles, but a lower expected total cost when vehicle fixed costs are taken into account.

Table 3. Computational results for stochastic problems

Test series	n	m	β_0	β_1	β_2	α	Expected duration of shortest route (minutes)	Expected duration of longest route (minutes)	Expected routing cost (BEF)	Expected lost interest due to early collections (BEF)	Expected lost interest due to late arrival at the depot (BEF)	Expected total cost (BEF)	Computation time (seconds)
1	28	8	1/25	1/28	1/19	0.000274	15	279	7660	2220	0	9880	170
		6	1/25	1/28	1/19	0.000274	77	230	5340	2168	0	7508	225
		5	1/25	1/28	1/19	0.000274	153	230	5310	2440	0	7750	120
2	44	12	1/40	1/47	1/26	0.000274	11	281	12975	3147	0	16122	250
		9	1/40	1/47	1/26	0.000274	35	299	12099	3612	3178	18889	184
		6	1/40	1/47	1/26	0.000274	132	264	11520	3815	3082	18417	300

When $n = 44$, route durations sometimes exceed 300 minutes (although their expected duration is always less than 300 minutes). In two scenarios, the expected penalty for lost interest due to late arrival at the depot is positive. When 6 or 9 vehicles are used, a tradeoff is made between starting collections relatively late to avoid interest loss due to early collections, and finishing late, thus risking late arrival at the depot. With 12 vehicles, collection routes are on the average shorter and no such tradeoff is necessary.

When closely examined and plotted on a map, our solutions seem realistic and make managerial sence. Also, the comparison between the deterministic and stochastic settings is quite revealing. In the stochastic case, more vehicles are required and the expected routing cost are higher. Since the optimal solutions are unknown we cannot compare our solutions to them, nor does it make much sense to compare them with lower bounds for the VRP since such bounds are notoriously weak, especially in the stochastic case.

6. Conclusion

We have analyzed in this paper a complex practical routing problem involving standard routing costs, as well as penalties for lost interest. In one version of the problem, stochastic travel times are also taken into account and the objective is then to minimize the total expected cost. Because of the intrinsic difficulty of this problem, and given the fact that typical problems are of medium or large scale, we have opted for a heuristic solution approach based on the Clarke and Wright algorithm. Problems derived from a real situation (a Belgian bank network) were solved under different scenarios. The results confirm the robustness of the proposed approach and provide decision makers with sufficient information to choose between different alternatives. Currently, collection routes are prepared by hand by the bank and there does not seem to exist any available commercial software capable of handling routing problems with interest penalties or stochastic components. We hope our work will be viewed as a useful step towards understanding and handling this difficult type of problem.

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Appendix : Glossary of main notation

Sets

$G = (V, A)$	:	a directed graph
$V = \{0, ..., n\}$	:	the vertex set (0 denotes the depot)
A	:	the arc set
$A' \subseteq A$		the set of arcs with random travel durations (arcs subject to congestion).

Constants for deterministic and stochastic problems

d_{ij}	:	distance associated with arc (i, j)
t_{ij}	:	travel duration on arc (i, j)
c_{ij}	:	travel cost on arc (i, j)
a_i	:	average daily sum collected at branch i
r_i	:	opening time of branch i
s_i	:	closing time of branch i
u_i	:	service time of branch i
v_i	:	deadline for visiting branch i
T	:	maximal allowed duration of any vehicle route
T'	:	cut-off time for same day crediting
f	:	vehicle fixed cost
α	:	unit penalty for lost interest
M	:	a very large constant.

Variables for deterministic and stochastic problems

x_{ij}	:	a binary variable equal to 1 of and only if arc (i, j) is used in the solution
y_i	:	the arrival time of the vehicle at branch i
m	:	the number of vehicles used in the solution.

Constants for stochastic problems

$a = \|A'\|$	:	the number of arcs subject to congestion
p	:	the probability that an arc is congested
β_0	:	the inverse of speed on arcs of $A \setminus A'$
β_1	:	the inverse of speed on uncongested arcs of A'
β_2	:	the inverse of speed on congested arcs of A'.

Variables for stochastic problems

$\bar{b}_\ell$	:	the expected value of vehicle content on route ℓ given that its duration exceeds T'
ξ_{ij}	:	a binary random variable equal to 1 if and only if arc $(i,j) \in A'$ is in a congested state.

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